Precision effective number of neutrinos N_{eff}:

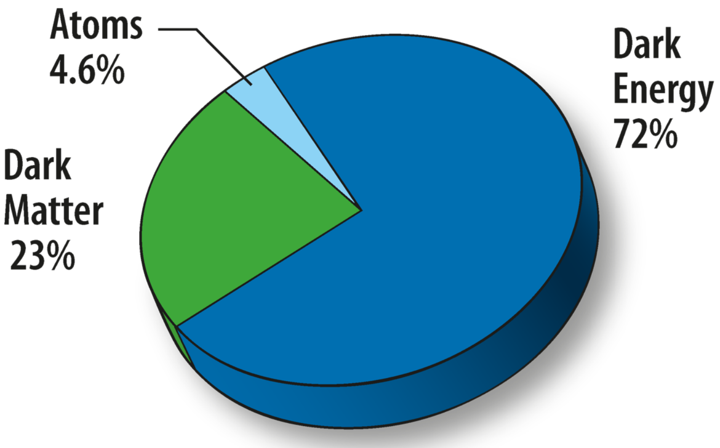
Yvonne Y. Y. Wong

UNSW Sydney

Vikram discussions on neutrino astrophysics, PRL, March 20, 2025

Concordance ΛCDM…

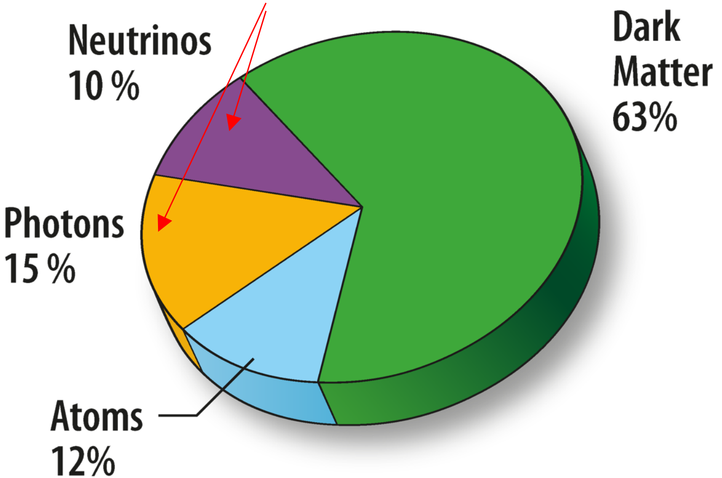
The simplest model consistent with most observations.



TODAY

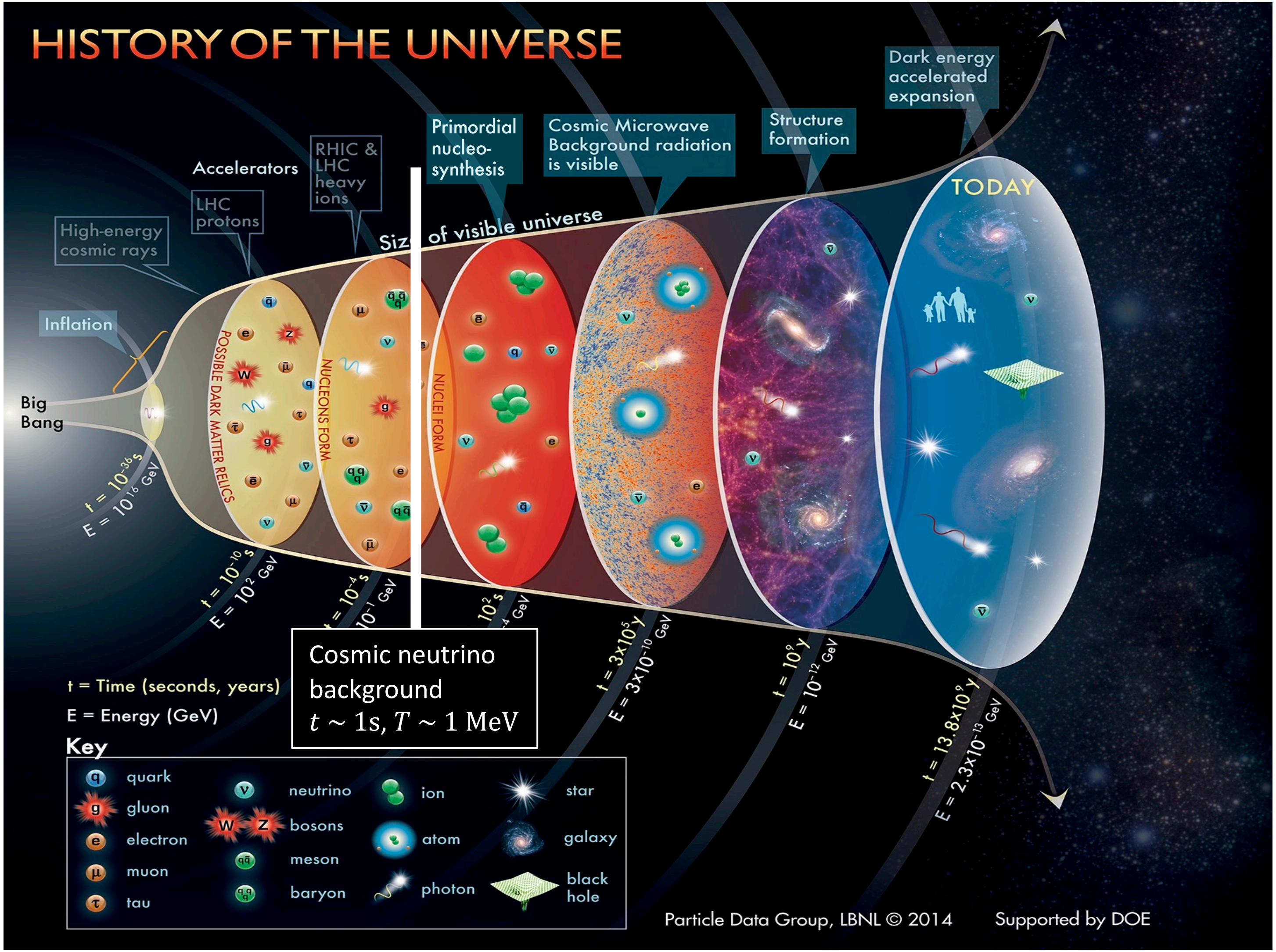
+ flat spatial geometry and initial conditions
consistent with single-field inflation

Neutrino-to-photon energy density ratio
fixed by SM physics, $\rho_\nu \approx 0.67\rho_\gamma$



13.7 BILLION YEARS AGO

(Universe 380,000 years old)



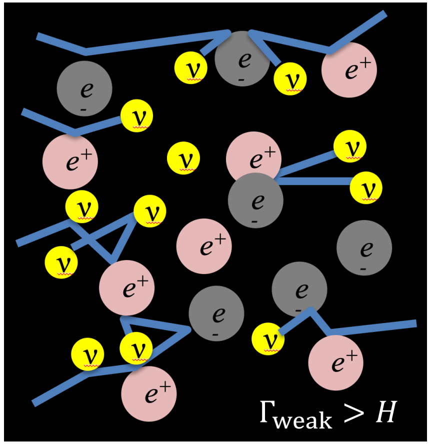
Cosmic neutrino background
$t \sim 1\text{s}, T \sim 1\ \text{MeV}$

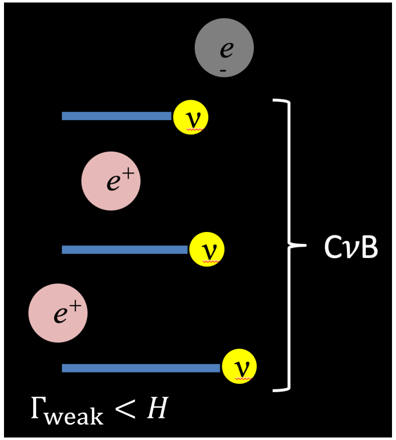
Interaction rate: $\Gamma_{\text{weak}} \sim G_F^2 T^5$

Expansion rate: $H \sim M_{\text{pl}}^{-2} T^2$

Formation of the CνB…

The CνB is formed when neutrinos decouple from the cosmic plasma.





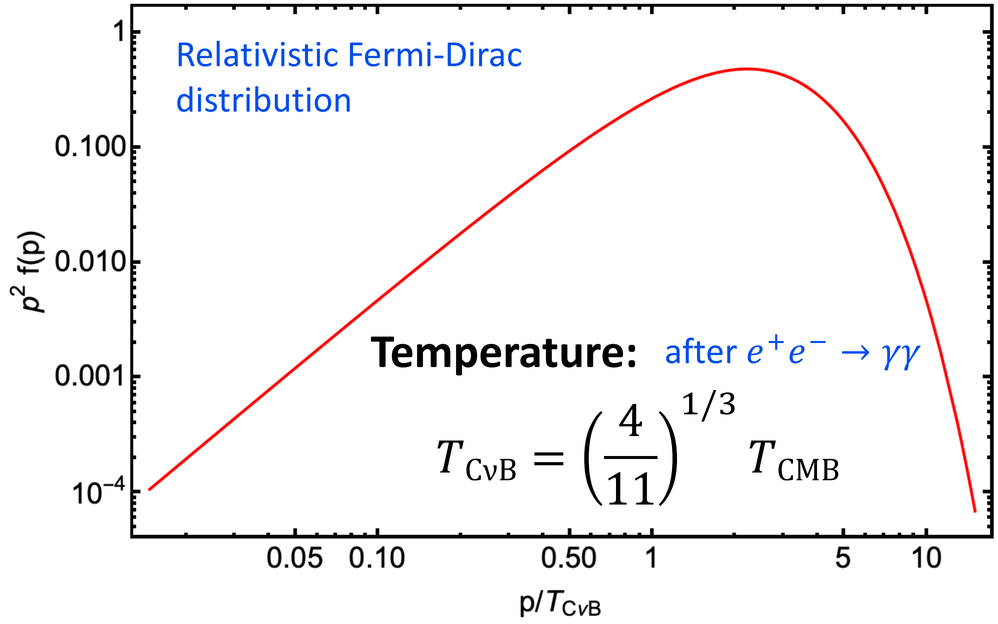
Neutrinos “free-stream” to infinity.

Above $T \sim 1$ MeV, even the Weak Interaction occurs efficiently enough to allow neutrinos to scatter off e^+e^- and other neutrinos, and attain thermodynamic equilibrium.

Below $T \sim 1$ MeV, expansion dilutes plasma, and reduces interaction rate: the universe becomes transparent to neutrinos.

The cosmic neutrino background…

Standard model predictions



Relativistic Fermi-Dirac distribution

Temperature: after $e^+e^- \to \gamma\gamma$

$$T_{C\nu B} = \left(\frac{4}{11}\right)^{1/3} T_{CMB}$$

Number density:

Per family of neutrinos +antineutrinos

$$n_{\nu,i} \simeq 110\ \text{cm}^{-3}$$

Energy density:

- Relativistic (if $T_{C\nu B} \gg m_\nu$):

$$\rho_{\nu,i} \simeq \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma \Rightarrow \frac{3\rho_{\nu,i}}{\rho_\gamma} \sim 0.68$$

- Non-rel (if $T_{C\nu B} \ll m_\nu$):

$$\Omega_{\nu,i} \simeq \frac{m_{\nu,i}}{93\ h^2\ \text{eV}}$$

Neutrino (hot) dark matter
→ cosmological neutrino mass bounds

Effective number of neutrinos…

A common practice is to express the neutrino-to-photon energy density ratio in terms of the **effective number of neutrino $\boldsymbol{N_{\mathrm{eff}}}$** parameter.

$$\sum_i \rho_{\nu,i} = N_{\mathrm{eff}} \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

Energy density in one thermalised species of massless fermions with 2 internal d.o.f. and temperature $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$.

The SM value is $N_{\mathrm{eff}}^{\mathrm{SM}} = 3.0440 \pm 0.0002$, for

- 3 families of neutrinos + antineutrinos
- A variety of %-level SM effects that alter **both** $\rho_{\nu,i}$ and ρ_γ from their naïve expectations.

Extending N_{eff} to light BSM thermal relics…

Any light (~sub-eV mass), feebly-interacting particle species produced by scattering in the early universe will **look sort of like a neutrino** as far as cosmology is concerned.

- E.g., light sterile neutrinos, thermal axions, …

- At leading order, these **light thermal relics** add to the SM neutrino energy density as if $N_{\text{eff}} \gtrsim 3$.
  → Re-interpret N_{eff} as the early-time non-photon radiation content:

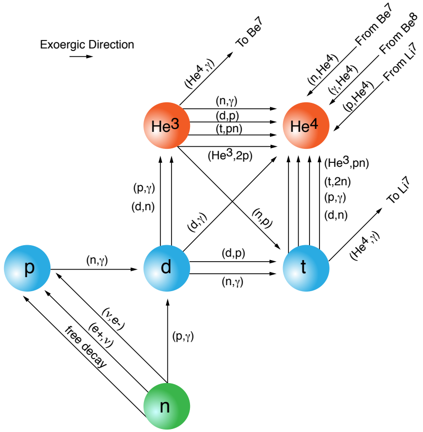
$$\sum_i \rho_{\nu,i} + \rho_{\text{other}} = N_{\text{eff}} \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

$$N_{\text{eff}} = N_{\text{eff}}^{\text{SM}} + \Delta N_{\text{eff}}$$

Why is N_{eff} interesting?

We **cannot detect the CνB** in the lab. But we can discern its presence from its impact on the events that take place after its formation.

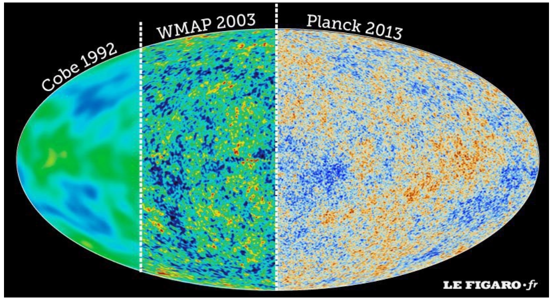
Light element abundances



Properties of the CνB probed:

N_{eff} (expansion rate)

CMB anisotropies

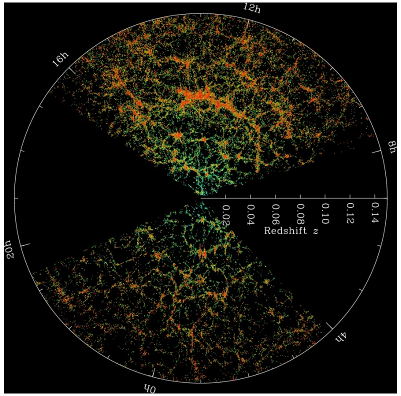


N_{eff} (expansion rate)

Interactions (free-streaming)

Lifetime (free-streaming)

Large-scale matter distribution



$\sum m_\nu$ (perturbation growth)

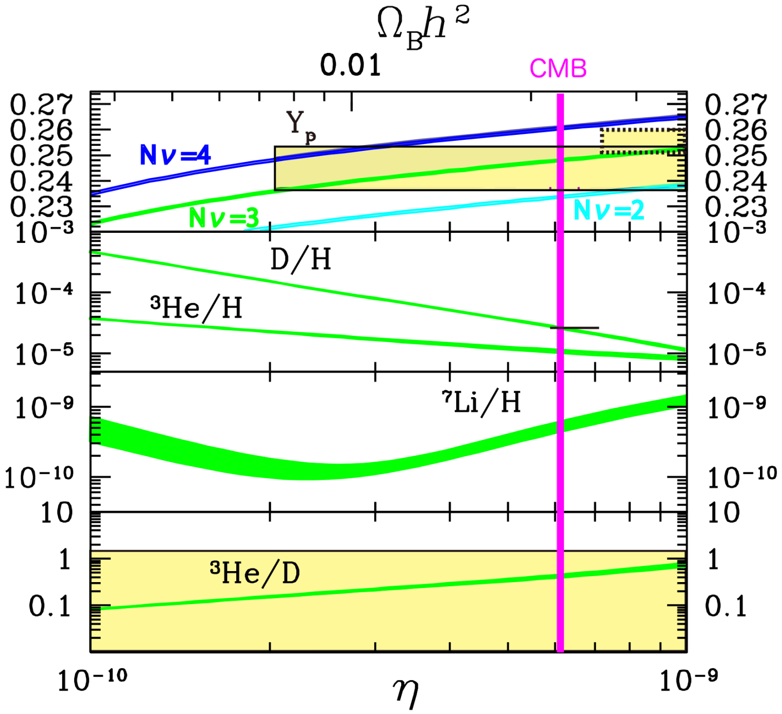
N_{eff} and nucleosynthesis...

Primordial nucleosynthesis takes place at $T \sim O(100) - O(10)$ keV, shortly after neutrino decoupling.

- Changing the expansion rate affects the production of **all** light elements.
- The largest effect is on He4, because
  - Almost all neutrons end up in He4.
  - The **neutron-to-proton ratio** depends strongly on how expansion affects the β-processes:

$$\nu_e + n \leftrightarrow p + e^-$$

$$\bar{\nu}_e + p \leftrightarrow n + e^+$$



Kawasaki, Kohri, Moroi & Takaesu 2018

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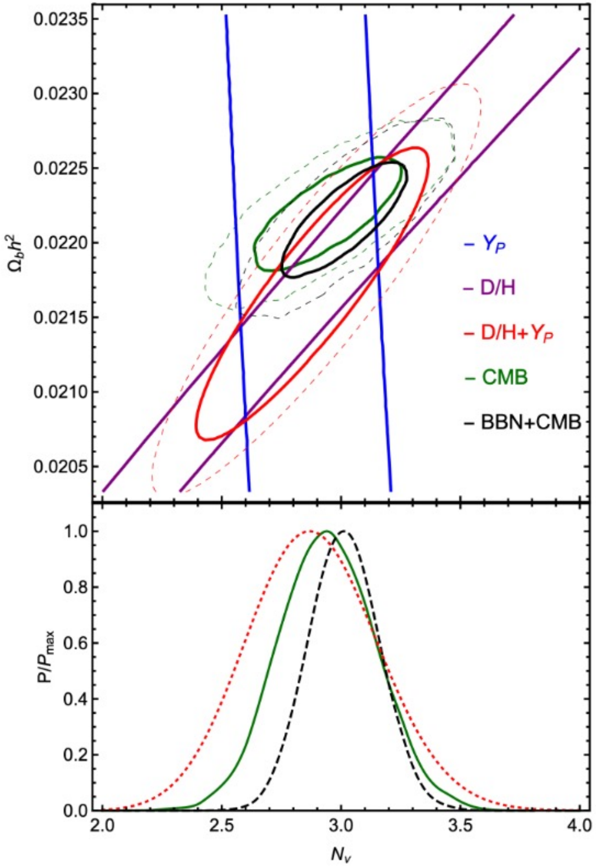


$$\boxed{N_{\mathrm{eff}} = 2.88 \pm 0.27 \text{ (68\% CL)}}$$

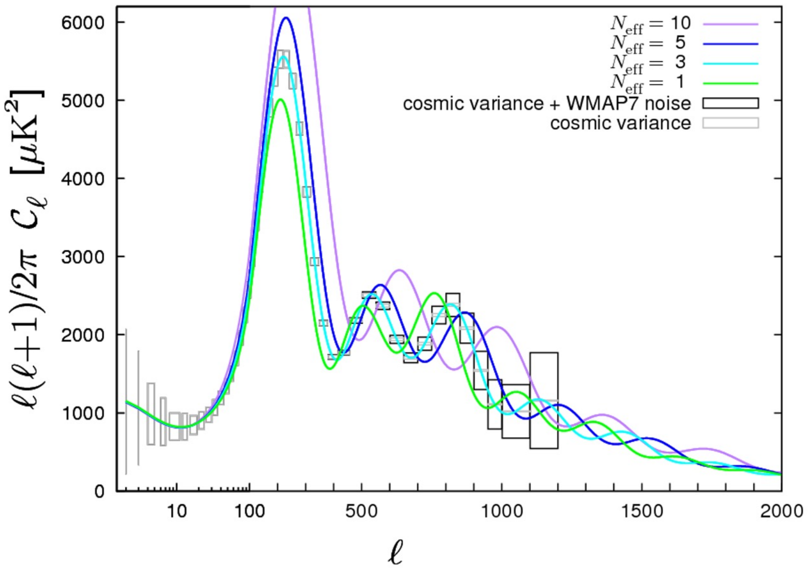
Pitrou, Coc, Uzan & Vangioni 2018

N_{eff} and the CMB anisotropies…

N_{eff} also affects the **expansion rate at recombination** ($T \sim 0.2\ eV$), observable in the CMB temperature power spectrum

- If you plug different values of N_{eff} into CAMB or CLASS, this is what you'll get.

"Naïve" signature



- But this is **not** the "real" effect of N_{eff}, because degeneracy with, e.g., the matter density ω_{m}, the Hubble parameter h, etc., can largely offset it.

N_{eff} and the CMB anisotropies…

N_{eff} also affects the **expansion rate at recombination** ($T{\sim}0.2$ eV), observable in the CMB temperature power spectrum

- Adjusting ω_{m} and h to match the first peak height and location, the irreducible signature of N_{eff} is in the damping tail.

Diffusion damping scale

$$r_d^2 \simeq (2\pi)^2 \int_0^{a_*} \frac{da}{a^3\sigma_T n_e H}\left[\frac{R^2+(16/15)(1+R)}{6(1+R)^2}\right]$$

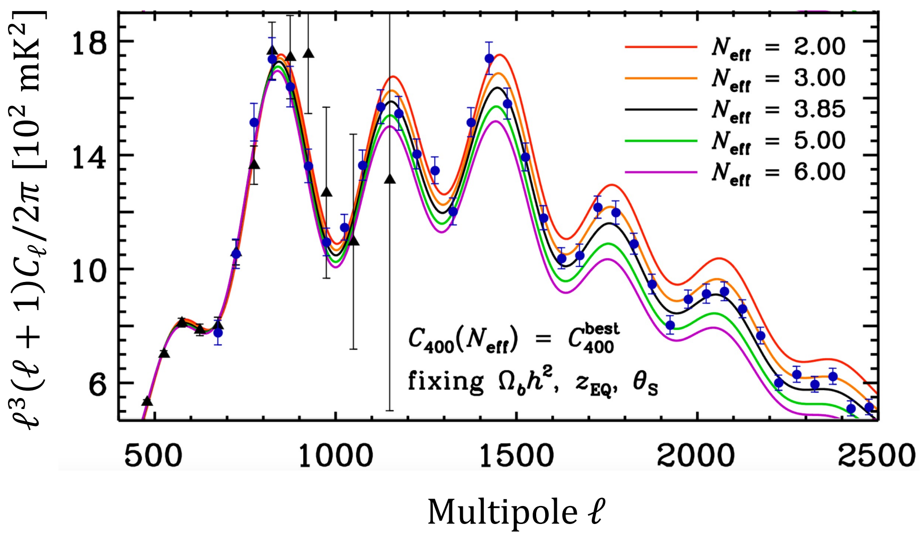
Thomson cross section

Free electron density

Hubble expansion

Baryon-to-photon density ratio

Hou et al. 2013

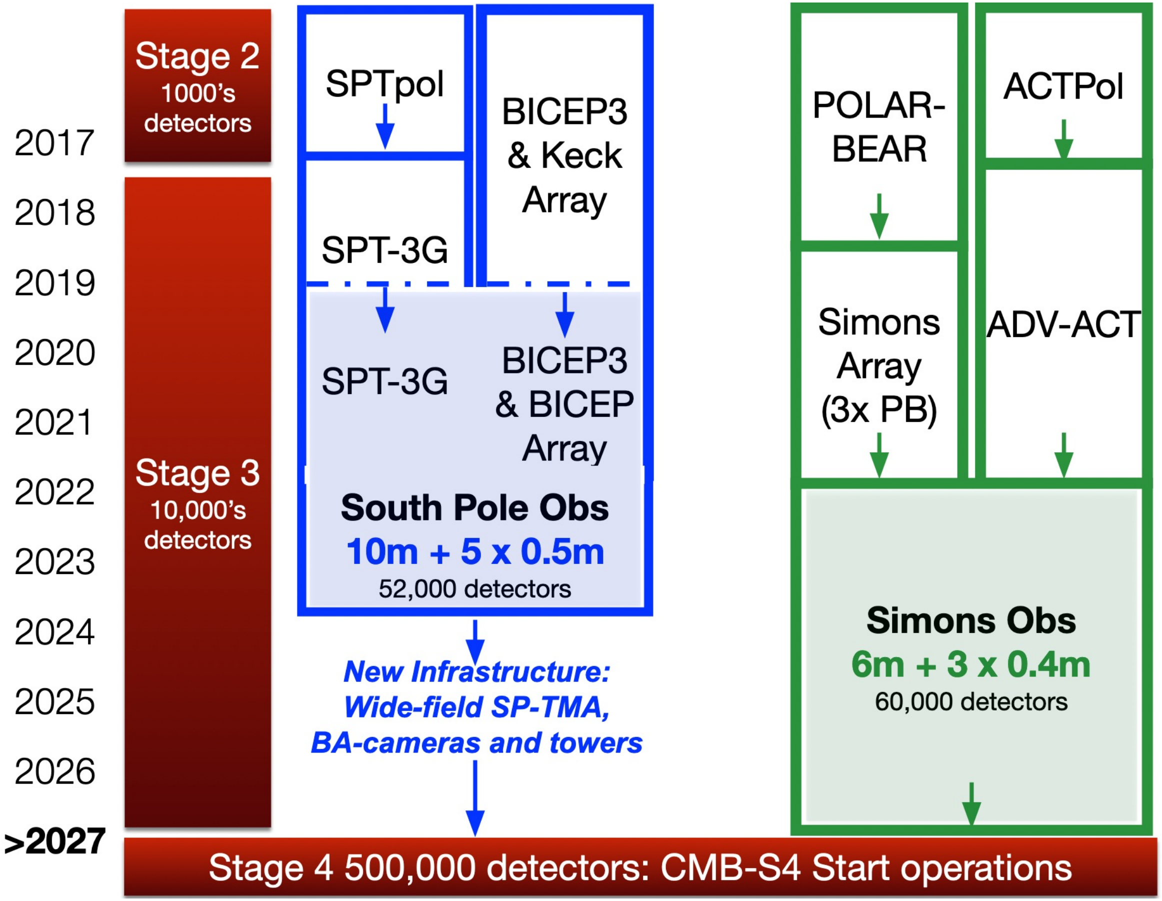


$$\boxed{N_{\text{eff}} = 2.99 \pm 0.34\ (95\%\ \text{CL})}$$

Planck TTTEEE +lowE+lensing+BAO; 7-parameters

Aghanim et al. [Planck] 2021

What to expect in the future?



2017
2018
2019
2020
2021
2022
2023
2024
2025
2026
>2027

Stage 2
1000's detectors

Stage 3
10,000's detectors

SPTpol

SPT-3G

BICEP3 & Keck Array

SPT-3G

BICEP3 & BICEP Array

South Pole Obs
10m + 5 x 0.5m
52,000 detectors

New Infrastructure: Wide-field SP-TMA, BA-cameras and towers

POLAR-BEAR

ACTPol

Simons Array (3x PB)

ADV-ACT

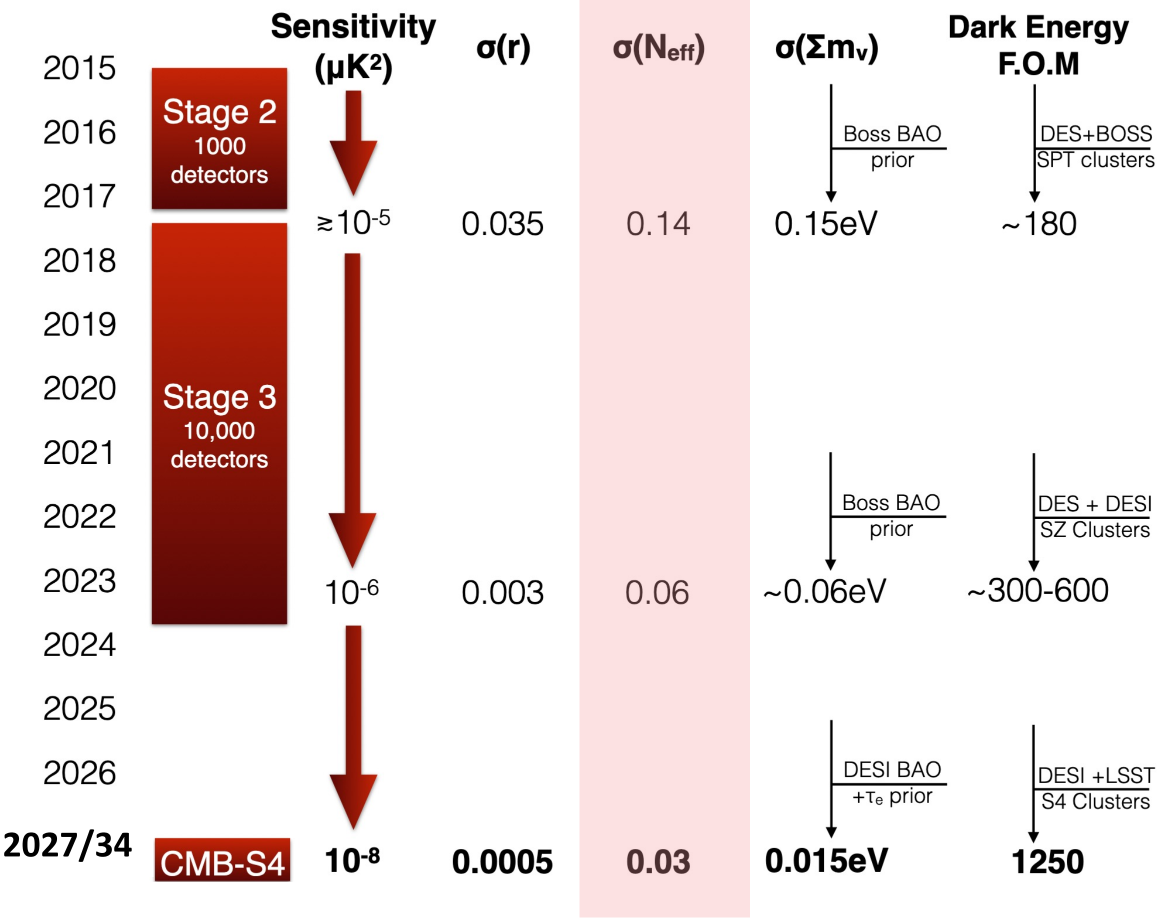
Simons Obs
6m + 3 x 0.4m
60,000 detectors

Stage 4 500,000 detectors: CMB-S4 Start operations

Future uncertain…

John Carlstrom

What to expect in the future?



		Sensitivity (μK²)	**σ(r)**	**σ(N_{eff})**	**σ(Σm_ν)**	**Dark Energy F.O.M**
2015						
2016	Stage 2 1000 detectors				Boss BAO prior	DES+BOSS SPT clusters
2017		≳10^{-5}	0.035	0.14	0.15eV	~180
2018						
2019						
2020	Stage 3 10,000 detectors					
2021						
2022					Boss BAO prior	DES + DESI SZ Clusters
2023		10^{-6}	0.003	0.06	~0.06eV	~300-600
2024						
2025						
2026					DESI BAO +τ_e prior	DESI +LSST S4 Clusters
2027/34	CMB-S4	**10^{-8}**	**0.0005**	**0.03**	**0.015eV**	**1250**

Future uncertain…

John Carlstrom

This talk…

Motivated by these future sensitivities to N_{eff}, we have dedicated a series of papers on computing the Standard-Model value $N_{\mathrm{eff}}^{\mathrm{SM}}$ **accurate to at least three decimal places**.

- What goes into the calculation of $N_{\mathrm{eff}}^{\mathrm{SM}} = 3.0440 \pm 0.0002$
- What other effects have been considered
- What remains to be done

Precision theoretical calculation of the Standard-Model $N_{\mathrm{eff}}^{\mathrm{SM}}$ …

The papers…

Towards a precision calculation of N_{eff} in the Standard Model:

1. The QED equation of state, *JCAP* 03 (2020) 003 [arXiv:1911.04504].

2. Neutrino decoupling in the presence of flavour oscillations and finite temperature QED, *JCAP* 04 (2021) 073 [axXiv:2012.02726]; Benchmark

3. Improved estimate of NLO contributions to collision integrals, *JCAP* 06 (2024) 032 [arXiv:2402.18481].

4. Impact of positronium formation, arXiv:2411.14091.

5. More collision integral @ NLO, in prep.

The team…

Leadership:

- Yvonne Wong, Marco Drewes, Michael Klasen (since 2024)

Our students and postdocs:

- UNSW: Giovanni Pierobon (4), Jack Bennett (1,2)
- UCLouvain: Yannis Georis (3, 4, 5), Gilles Buldgen (1,2)
- Münster: Adrian Finke (5), Luca Wiggering (3,5)

Friends with special tools:

- IFIC Valencia: Sergio Pastor (2), Stefano Gariazzo (2), Pablo de Salas (2)
- TU Munich: Tobias Binder (4)

Also recent works by others…

Comparable to our benchmark paper 2 including correction from our paper1:

- Akita & Yamaguchi, *JCAP 08 (2020) 012* [arXiv:2005.07047].
- Froustey, Pitrou & Volpe, *JCAP 12 (2020) 015* [arXiv:2008.0107].

Collision integral @ NLO; comparable to our papers 3 and 5:

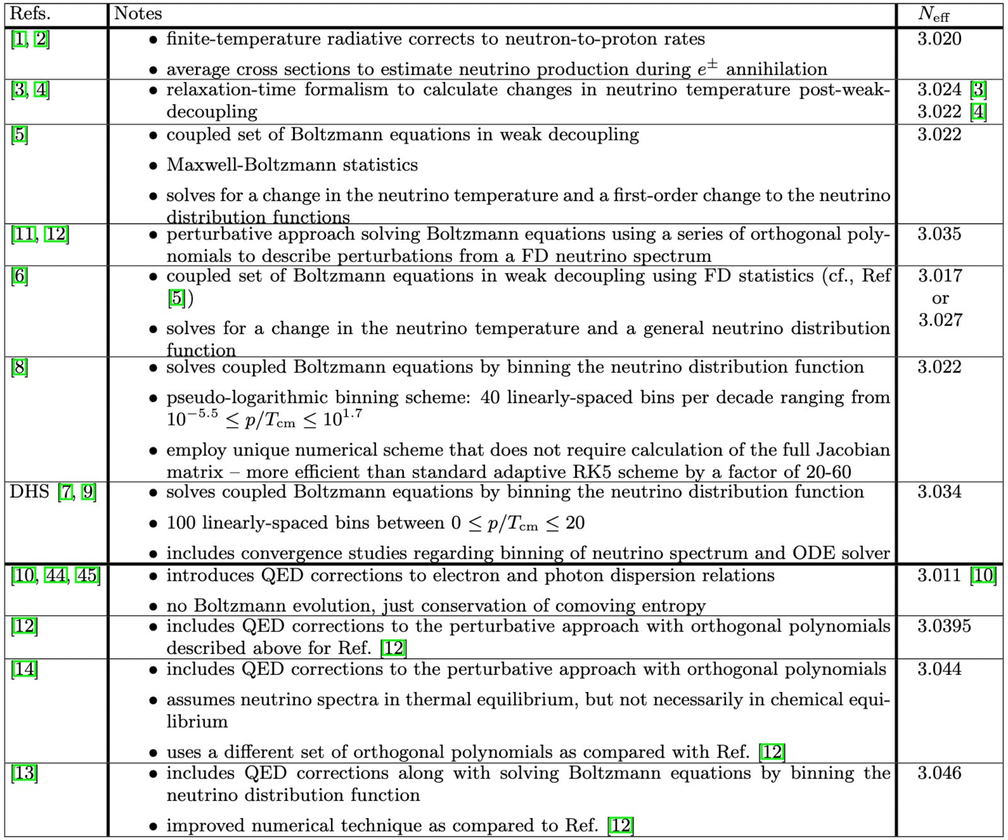
- Cielo, Escudero, Magano & Pisanti, *PRD 108 (2023) L121301* [arXiv:2306.05460].
- Jackson & Laine, *JHEP 05 (2024) 089* [arXiv: 2312.07015]; arXiv: 2412.03958.

Anisotropic neutrino forward scattering:

- Hansen, Shalgar & Tamborra, *JCAP 07 (2021) 017* [arXiv:2012.03948].

Precision $N_{\rm eff}^{\rm SM}$: an old subject…

1982



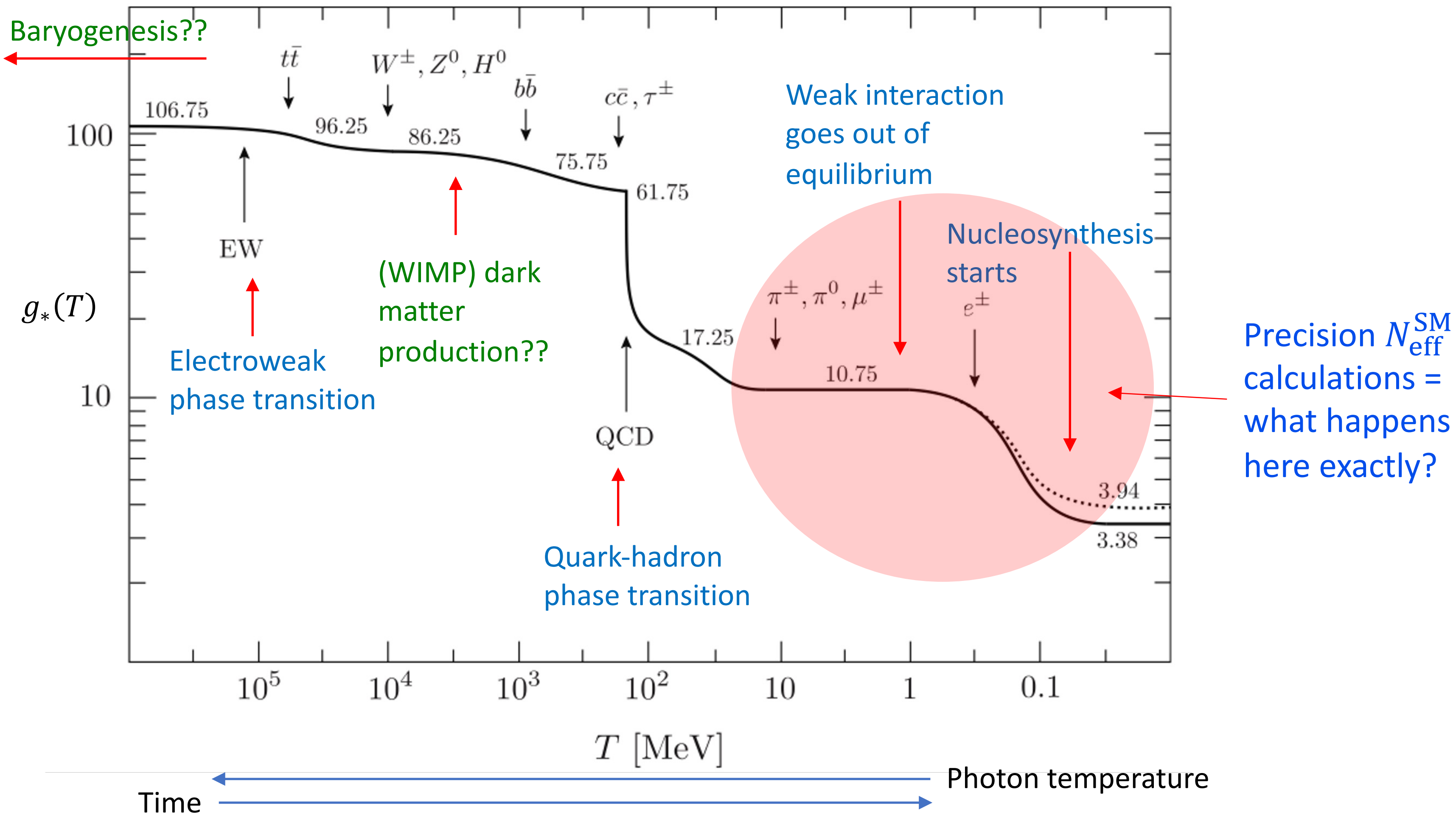
Refs.	Notes	$N_{\rm eff}$
[1, 2]	• finite-temperature radiative corrects to neutron-to-proton rates • average cross sections to estimate neutrino production during $e^{\pm}$ annihilation	3.020
[3, 4]	• relaxation-time formalism to calculate changes in neutrino temperature post-weak-decoupling	3.024 [3] 3.022 [4]
[5]	• coupled set of Boltzmann equations in weak decoupling • Maxwell-Boltzmann statistics • solves for a change in the neutrino temperature and a first-order change to the neutrino distribution functions	3.022
[11, 12]	• perturbative approach solving Boltzmann equations using a series of orthogonal polynomials to describe perturbations from a FD neutrino spectrum	3.035
[6]	• coupled set of Boltzmann equations in weak decoupling using FD statistics (cf., Ref [5]) • solves for a change in the neutrino temperature and a general neutrino distribution function	3.017 or 3.027
[8]	• solves coupled Boltzmann equations by binning the neutrino distribution function • pseudo-logarithmic binning scheme: 40 linearly-spaced bins per decade ranging from $10^{-5.5} \leq p/T_{\rm cm} \leq 10^{1.7}$ • employ unique numerical scheme that does not require calculation of the full Jacobian matrix – more efficient than standard adaptive RK5 scheme by a factor of 20-60	3.022
DHS [7, 9]	• solves coupled Boltzmann equations by binning the neutrino distribution function • 100 linearly-spaced bins between $0 \leq p/T_{\rm cm} \leq 20$ • includes convergence studies regarding binning of neutrino spectrum and ODE solver	3.034
[10, 44, 45]	• introduces QED corrections to electron and photon dispersion relations • no Boltzmann evolution, just conservation of comoving entropy	3.011 [10]
[12]	• includes QED corrections to the perturbative approach with orthogonal polynomials described above for Ref. [12]	3.0395
[14]	• includes QED corrections to the perturbative approach with orthogonal polynomials • assumes neutrino spectra in thermal equilibrium, but not necessarily in chemical equilibrium • uses a different set of orthogonal polynomials as compared with Ref. [12]	3.044
[13]	• includes QED corrections along with solving Boltzmann equations by binning the neutrino distribution function • improved numerical technique as compared to Ref. [12]	3.046

2005

Grohs et al. 2016

3.052??

g_* of the standard model of particle physics:



Baryogenesis??

Electroweak phase transition

(WIMP) dark matter production??

Quark-hadron phase transition

Weak interaction goes out of equilibrium

Nucleosynthesis starts

Precision $N_{\text{eff}}^{\text{SM}}$ calculations = what happens here exactly?

Photon temperature

Time

Particle content at $0.1 < T < 10$ MeV…

The particle content and interactions at $0.1 < T < 10$ MeV determine the properties of the CνB.

- **QED plasma**: $e^{\pm}, \gamma$
- **3 families of $\boldsymbol{\nu + \bar{\nu}}$**: $\nu_e, \bar{\nu}_e,$ $\nu_\mu, \bar{\nu}_\mu,$ $\nu_\tau, \bar{\nu}_\tau$

EM interactions (always in equilibrium @ $0.1 < T < 10$ MeV):

$e^+e^- \leftrightarrow \gamma\gamma$

$e^+e^- \leftrightarrow e^+e^-$

$e^{\pm}e^{\mp} \leftrightarrow e^{\pm}e^{\mp}$

$e^{\pm}e^{\pm} \leftrightarrow e^{\pm}e^{\pm}$

$\gamma e^{\pm} \leftrightarrow \gamma e^{\pm}$

Coupled @ $T > O(1)$MeV

$\nu_\alpha e^{\pm} \leftrightarrow \nu_\alpha e^{\pm}$

$\nu_\alpha \bar{\nu}_\alpha \leftrightarrow e^+e^-$

Weak interactions (in equilibrium @ $T > O(1)$MeV)

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$\nu_\alpha \bar{\nu}_\beta \leftrightarrow \nu_\alpha \bar{\nu}_\beta$

$\bar{\nu}_\alpha \bar{\nu}_\beta \leftrightarrow \bar{\nu}_\alpha \bar{\nu}_\beta$

$\alpha, \beta = e, \mu, \tau$

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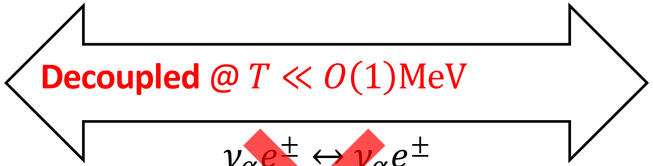

Decoupled @ $T \ll O(1)$MeV

$$\xcancel{\nu_\alpha e^{\pm} \leftrightarrow \nu_\alpha e^{\pm}}$$
$$\xcancel{\nu_\alpha \bar{\nu}_\alpha \leftrightarrow e^+e^-}$$

Weak interactions (**not** in equilibrium @ $T \ll O(1)$MeV)

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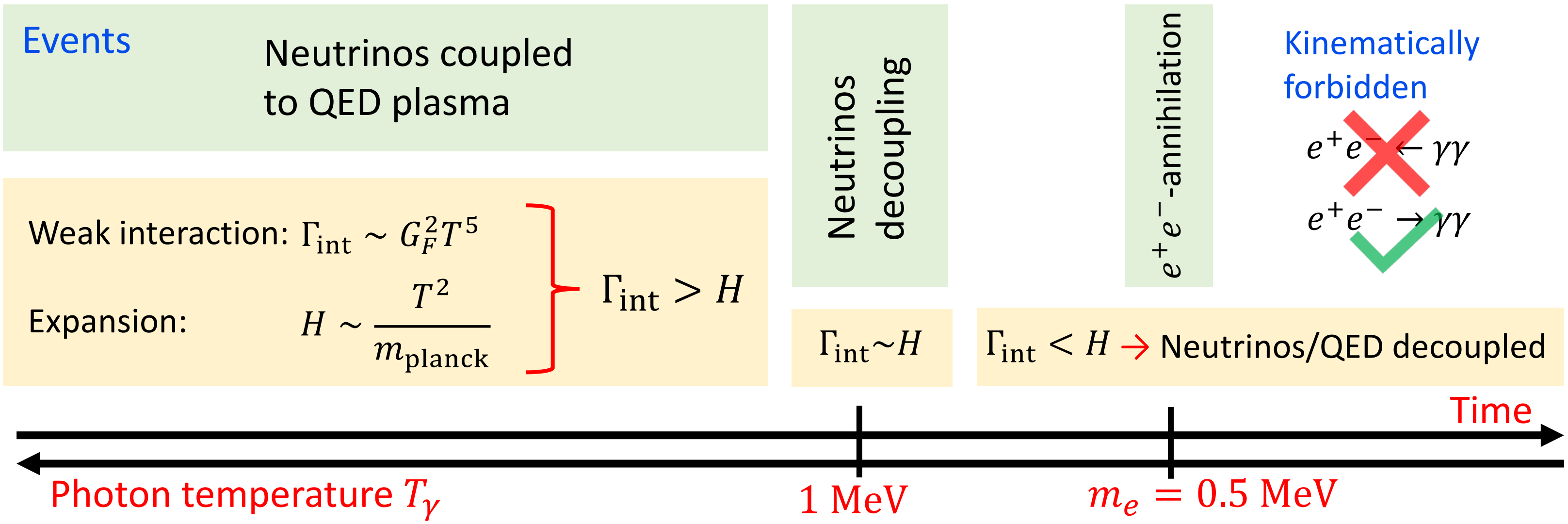
Thermal history of neutrinos...



Events

Neutrinos coupled to QED plasma

Neutrinos decoupling

e^+e^--annihilation

Kinematically forbidden

$\cancel{e^+e^- \leftarrow \gamma\gamma}$

$e^+e^- \rightarrow \gamma\gamma$ ✓

Weak interaction: $\Gamma_{\text{int}} \sim G_F^2 T^5$

Expansion: $H \sim \dfrac{T^2}{m_{\text{planck}}}$

$\Big\} \; \Gamma_{\text{int}} > H$

$\Gamma_{\text{int}} \sim H$

$\Gamma_{\text{int}} < H$ → Neutrinos/QED decoupled

Time

Photon temperature T_γ

$1\ \text{MeV}$

$m_e = 0.5\ \text{MeV}$

Neutrino temperature

$$T_\nu = T_\gamma$$

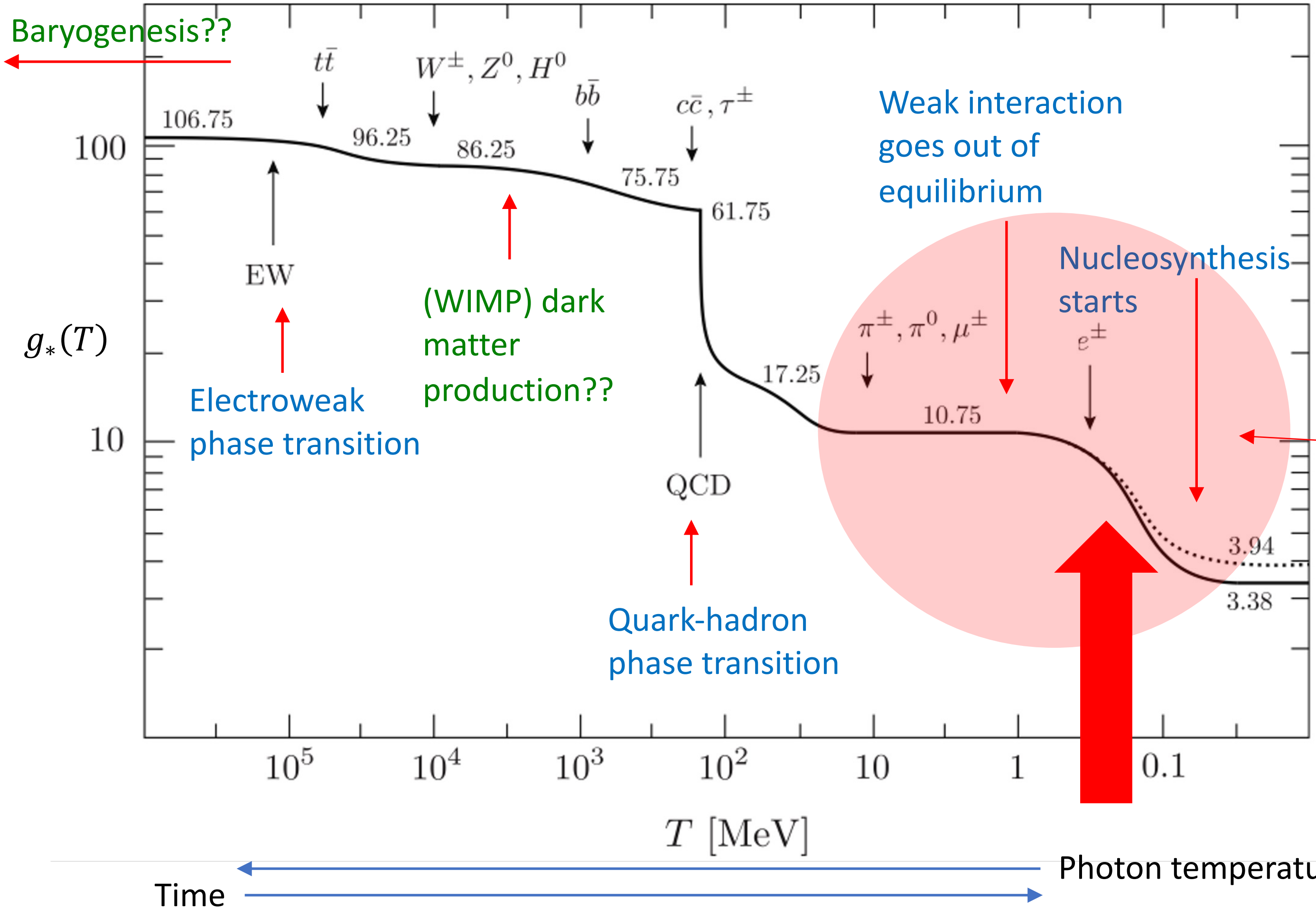
$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

Phase space distribution

$$f_\nu(p) \approx \frac{1}{\exp[p/T_\nu] + 1}$$

Relativistic Fermi-Dirac

g_* of the standard model of particle physics:



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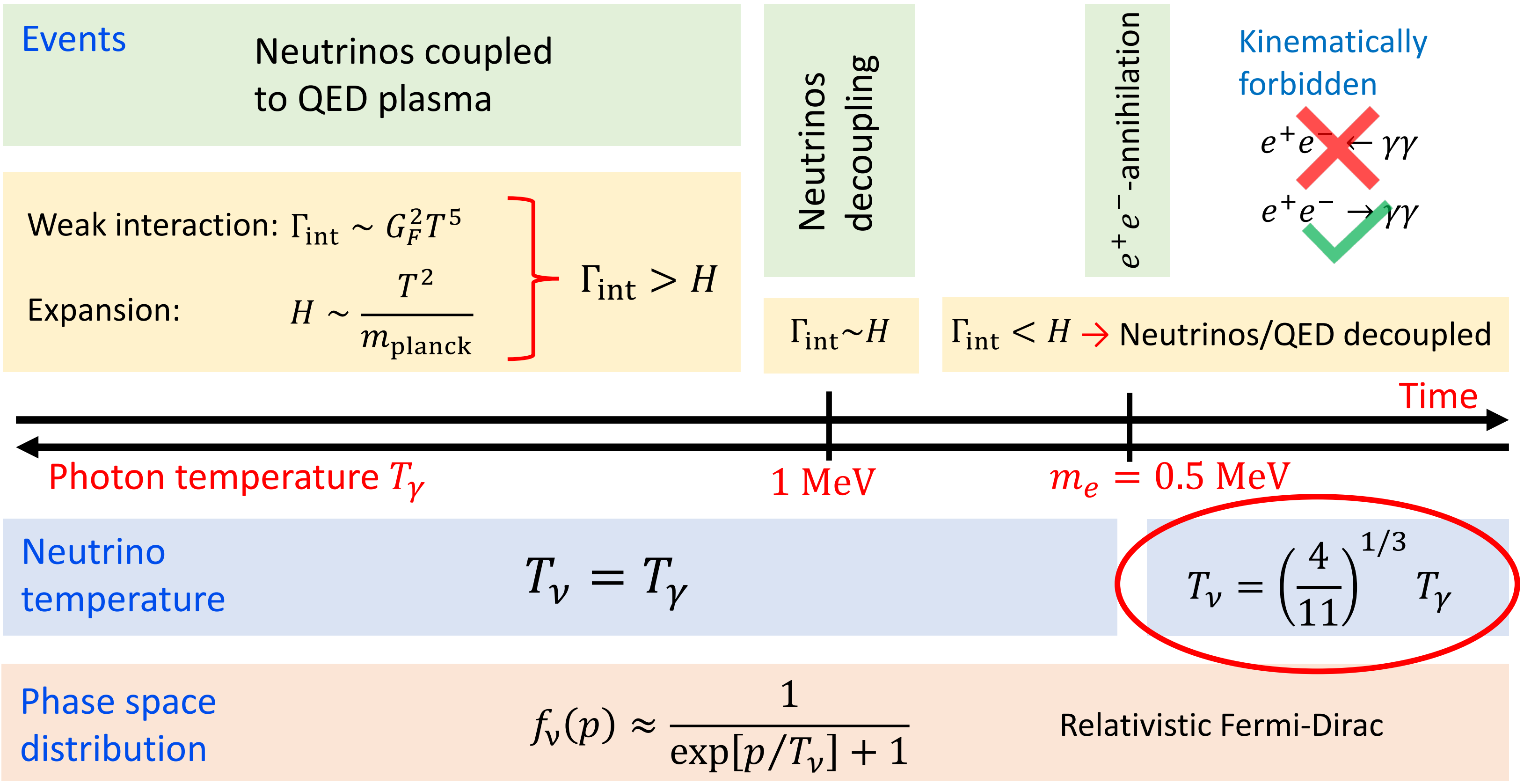

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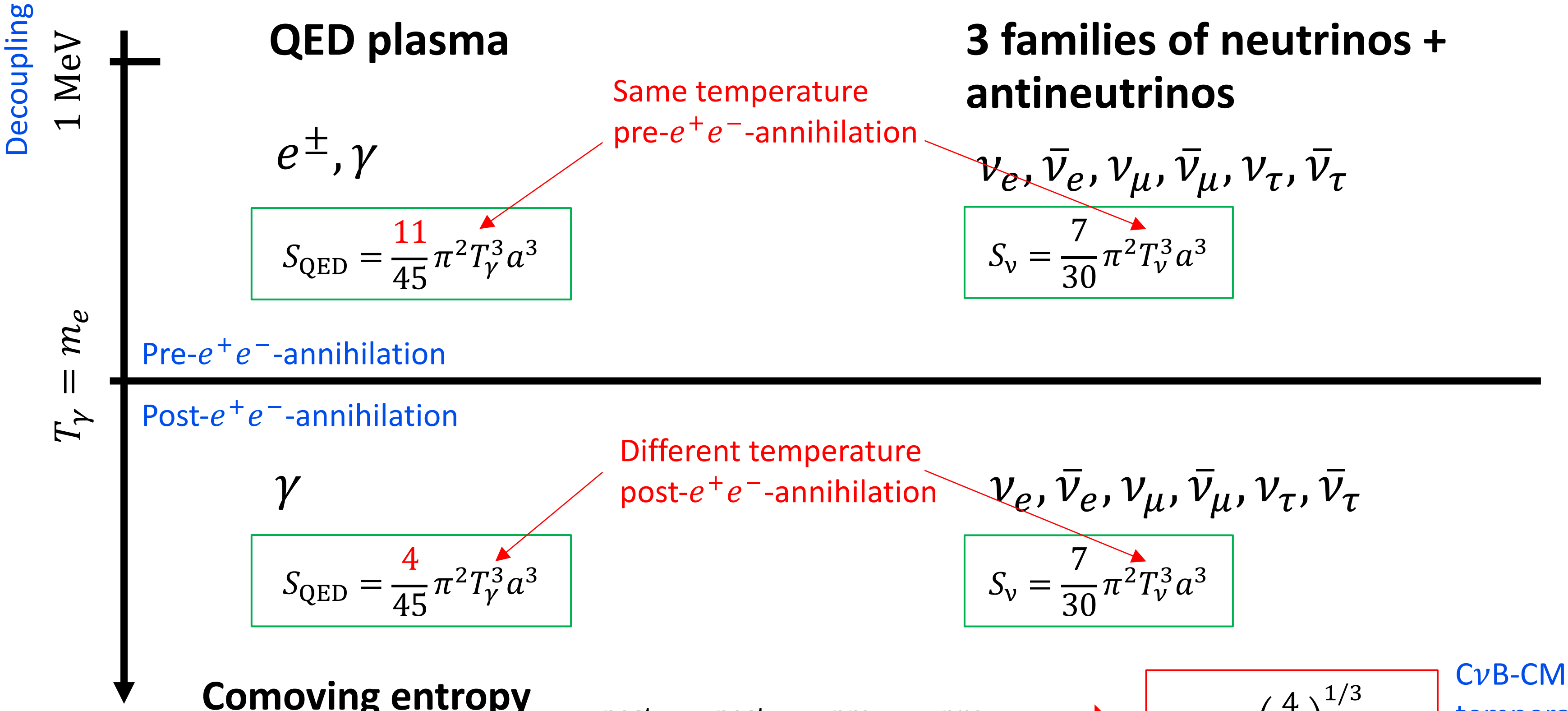
Relativistic Fermi-Dirac

T_ν from comoving entropy conservation…



QED plasma

3 families of neutrinos + antineutrinos

Same temperature pre-e^+e^--annihilation

$e^\pm, \gamma$

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

$$\boxed{S_{\mathrm{QED}} = \frac{11}{45}\pi^2 T_\gamma^3 a^3}$$

$$\boxed{S_\nu = \frac{7}{30}\pi^2 T_\nu^3 a^3}$$

Pre-e^+e^--annihilation

Post-e^+e^--annihilation

Different temperature post-e^+e^--annihilation

γ

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

$$\boxed{S_{\mathrm{QED}} = \frac{4}{45}\pi^2 T_\gamma^3 a^3}$$

$$\boxed{S_\nu = \frac{7}{30}\pi^2 T_\nu^3 a^3}$$

Comoving entropy conservation:

$$S_{\mathrm{QED}}^{\mathrm{post}} + S_\nu^{\mathrm{post}} = S_{\mathrm{QED}}^{\mathrm{pre}} + S_\nu^{\mathrm{pre}} \Longrightarrow \boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma}$$

CνB-CMB temperature relation

Naïve $N_{\mathrm{eff}}^{\mathrm{SM}}$ …

Taking $T_\nu = (4/11)^{1/3} T_\gamma$, we expect the **neutrino-to-photon energy density ratio** at low temperatures ($T_\gamma \ll m_e$) to be:

Fermi-Dirac statistics

$$\sum_i \rho_{\nu,i} = 3 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

Three families

(Temperature ratio)^4

Precision $N_{\mathrm{eff}}^{\mathrm{SM}}$ …

In reality, the neutrino-to-photon energy density ratio is about a percent higher than the naïve estimate:

Fermi-Dirac statistics

$$\sum_i \rho_{\nu,i} = N_{\mathrm{eff}} \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

$N_{\mathrm{eff}} = 3.0440 \pm 0.0002$

(Temperature ratio)^4

Conventionally, we absorb all corrections into the N_{eff} parameter.

Deviations from 3 due to:

- Non-instantaneous neutrino decoupling
- Neutrino flavour oscillations

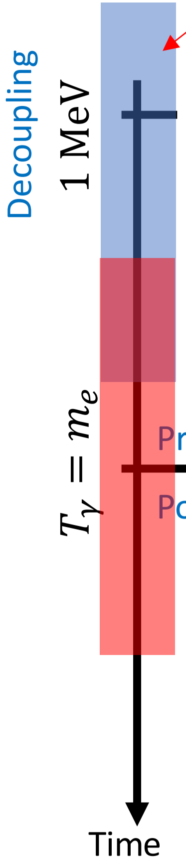
Corrections to ρ_ν

- Non-relativistic electron gas across neutrino decoupling
- Non-ideal gas corrections to the photon/electron plasma

Corrections to ρ_γ

Correction to ρ_ν…

Deviations, or what's wrong with this picture?



Neither neutrino decoupling nor annihilation is instantaneous → some energy of the annihilation is transferred to the high-energy neutrinos still coupled.

QED plasma

$e^\pm, \gamma$

$$S_{\text{QED}} = \frac{11}{45}\pi^2 T_\gamma^3 a^3$$

3 families of neutrinos + antineutrinos

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

$$S_\nu = \frac{7}{30}\pi^2 T_\nu^3 a^3$$

Pre-e^+e^--annihilation

Post-e^+e^--annihilation

Out-of-equilibrium neutrinos → entropy conservation strictly does not hold.

γ

$$S_{\text{QED}} = \frac{4}{45}\pi^2 T_\gamma^3 a^3$$

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

$$S_\nu = \frac{7}{30}\pi^2 T_\nu^3 a^3$$

Comoving entropy conservation:

$$S_{\text{QED}}^{\text{post}} + S_\nu^{\text{post}} = S_{\text{QED}}^{\text{pre}} + S_\nu^{pre} \Rightarrow T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

CνB-CMB temperature relation

Tracking non-instantaneous decoupling…

The effect of an out-of-equilibrium interaction on a particle species can be tracked using the Boltzmann equation.

f_1 = Phase space density of the particle species of interest

Collision term

$$\frac{\partial f_1}{\partial t} = -\{f_1, H\} + C[f_1]$$

Hamiltonian for particle propagation

- **The collision term** for e.g., $1 + 2 \rightarrow 3 + 4$

9D phase space integral

Energy-momentum conservation

Matrix element

$$C[f_1] = \frac{1}{2E_1} \int \prod_{i=2}^{4} \frac{d^3 p_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^4 (P_1 + P_2 - P_3 - P_4) |M|^2$$
$$\times [f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)]$$

Quantum statistical factors

Tracking decoupling including oscillations…

Tracking neutrino decoupling is complicated by **neutrino oscillations**.

- We promote the classical Boltzmann equation for the phase space density to a quantum kinetic equation (QKE) for the density matrix of the neutrino ensemble.

Boltzmann

$$\frac{\partial f_1}{\partial t} = -\{f_1, H\} + C[f_1]$$

Quantum kinetic equation

Collision term

$$\boxed{\frac{\partial \hat{\rho}_1}{\partial t} = -\frac{1}{i\hbar}\left[\hat{\rho}_1, \widehat{H}\right] + \hat{C}[\hat{\rho}]}$$

e.g., Sigl & Raffelt 1993

Density matrix (momentum-dependent)

$$\hat{\rho} = \begin{pmatrix} \rho_{ee} & \rho_{e\mu} & \rho_{e\tau} \\ \rho_{\mu e} & \rho_{\mu\mu} & \rho_{\mu\tau} \\ \rho_{\tau e} & \rho_{\tau\mu} & \rho_{\tau\tau} \end{pmatrix}$$

Diagonal ~ occupation numbers
Off-diagonal ~ oscillation phases

Hamiltonian

$$\widehat{H} = \frac{1}{2p} U \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} U^{\dagger} + \hat{V}_{\text{matter}}$$

Vacuum oscillations + matter effects (“thermal masses”)

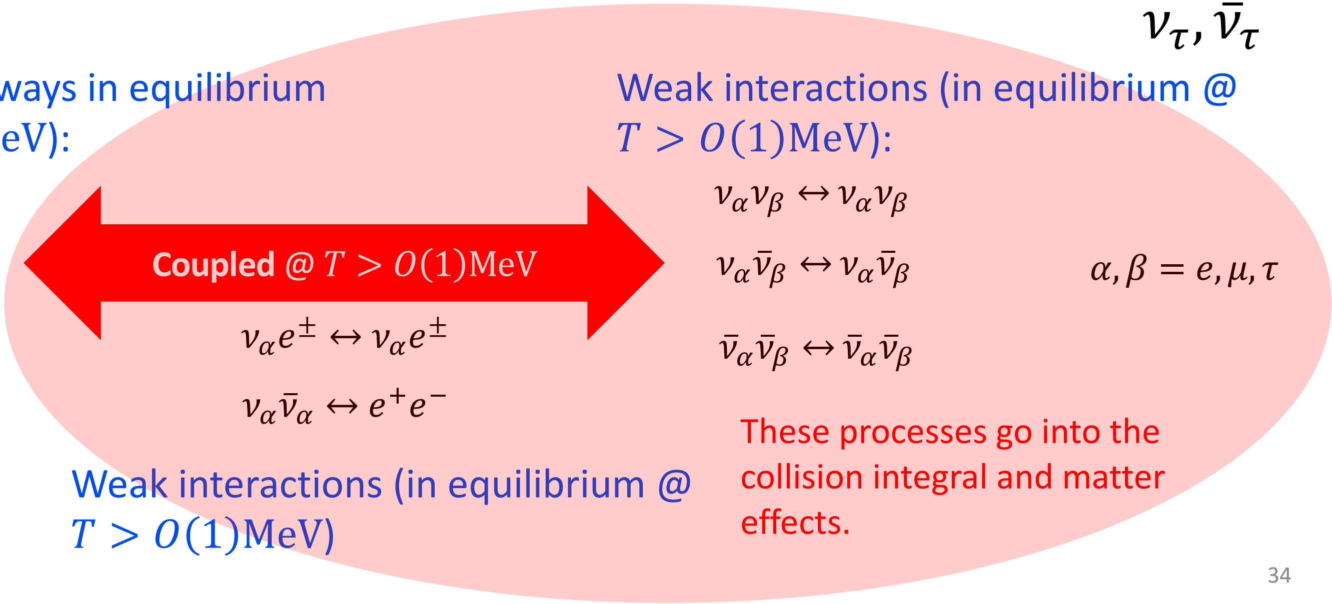
Interactions at $0.1 < T < 10$ MeV…

The particle content and interactions at $0.1 < T < 10$ MeV determine the properties of the CνB.

- **QED plasma**: $e^{\pm}, \gamma$
- **3 families of $\boldsymbol{\nu + \bar{\nu}}$**: $\nu_e, \bar{\nu}_e,$ $\nu_\mu, \bar{\nu}_\mu,$ $\nu_\tau, \bar{\nu}_\tau$

EM interactions (always in equilibrium @ $0.1 < T < 10$ MeV):

$$e^+e^- \leftrightarrow \gamma\gamma$$
$$e^+e^- \leftrightarrow e^+e^-$$
$$e^{\pm}e^{\mp} \leftrightarrow e^{\pm}e^{\mp}$$
$$e^{\pm}e^{\pm} \leftrightarrow e^{\pm}e^{\pm}$$
$$\gamma e^{\pm} \leftrightarrow \gamma e^{\pm}$$



Coupled @ $T > O(1)\mathrm{MeV}$

$$\nu_\alpha e^{\pm} \leftrightarrow \nu_\alpha e^{\pm}$$
$$\nu_\alpha \bar{\nu}_\alpha \leftrightarrow e^+e^-$$

Weak interactions (in equilibrium @ $T > O(1)\mathrm{MeV}$)

Weak interactions (in equilibrium @ $T > O(1)\mathrm{MeV}$):

$$\nu_\alpha \nu_\beta \leftrightarrow \nu_\alpha \nu_\beta$$
$$\nu_\alpha \bar{\nu}_\beta \leftrightarrow \nu_\alpha \bar{\nu}_\beta \qquad \alpha, \beta = e, \mu, \tau$$
$$\bar{\nu}_\alpha \bar{\nu}_\beta \leftrightarrow \bar{\nu}_\alpha \bar{\nu}_\beta$$

These processes go into the collision integral and matter effects.

Collision integrals @ LO…

Weak annihilation and scattering rates are currently computed to leading order in G_F, i.e., $O(G_F^2)$.

- These have been **incorporated in the quantum kinetic equations**, which are solved with full momentum dependence plus quantum statistics in our benchmark paper 2 using a dedicated decoupling code `FortEPiaNO`. Gariazzo, de Salas & Pastor 2019

- What about higher-order contributions?

Collision integrals @ NLO…

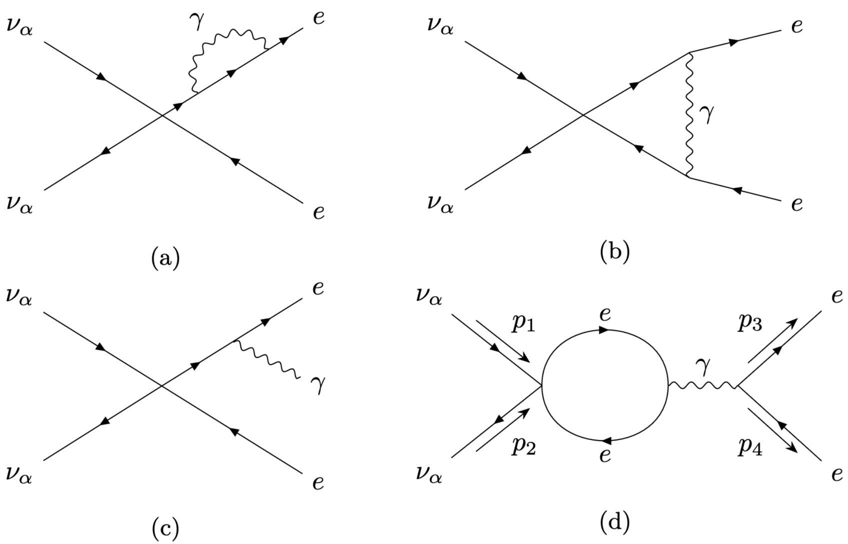
There has been some recent interest in computing QED corrections ($T = 0$ + finite-temperature) to the **weak annihilation and scattering rates**.

- Mainly motivated by claims of a large $|\delta N_{\rm eff}| \sim 0.001$ from these corrections.

  Cielo, Escudero, Mangano & Pisanti 2023

- However, a much smaller $|\delta N_{\rm eff}| \lesssim 10^{-5}$ was found in independent works.

  Jackson & Laine 2023, 2024
  Drewes, Georis, Klasen, Wiggering & Y^{3}W 2024



Probably the fairest thing to say at this stage is that there is as yet no complete calculation of the effect of NLO weak rates on N_{eff}.

	Correction	First principles	Finite m_e	Quantum stats	Solve QKEs
Paper 2 (benchmark)	Type a only	No; modelled as a thermal electron mass	Yes	Yes	Yes
Cielo et al.	Types a-c only	No; mapped from stellar plasma calculations	Yes	No	In some approximation
Jackson & Laine	All	Yes	No; Hard Thermal Loop approximation	Yes	No
Paper 3	Type d only	Yes	Yes	Yes	Damping approximation

- Paper 5 (in prep) will hopefully fix that. Stay tuned!

Correction to ρ_γ…

Deviations, or what’s wrong with this picture?

Decoupling

1 MeV

QED plasma

$e^\pm, \gamma$

$$S_{\text{QED}} = \frac{11}{45}\pi^2 T_\gamma^3 a^3$$

At $T \sim 1$ MeV, $e^\pm$ are not ultra-relativistic, nor is the QED plasma an ideal gas to justify this expression.

3 families of neutrinos + antineutrinos

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

$$S_\nu = \frac{7}{30}\pi^2 T_\nu^3 a^3$$

$T_\gamma = m_e$

Pre-e^+e^--annihilation

Post-e^+e^--annihilation

γ

$$S_{\text{QED}} = \frac{4}{45}\pi^2 T_\gamma^3 a^3$$

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

$$S_\nu = \frac{7}{30}\pi^2 T_\nu^3 a^3$$

Time

Comoving entropy conservation:

$$S_{\text{QED}}^{\text{post}} + S_\nu^{\text{post}} = S_{\text{QED}}^{\text{pre}} + S_\nu^{\text{pre}} \Rightarrow \boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma}$$

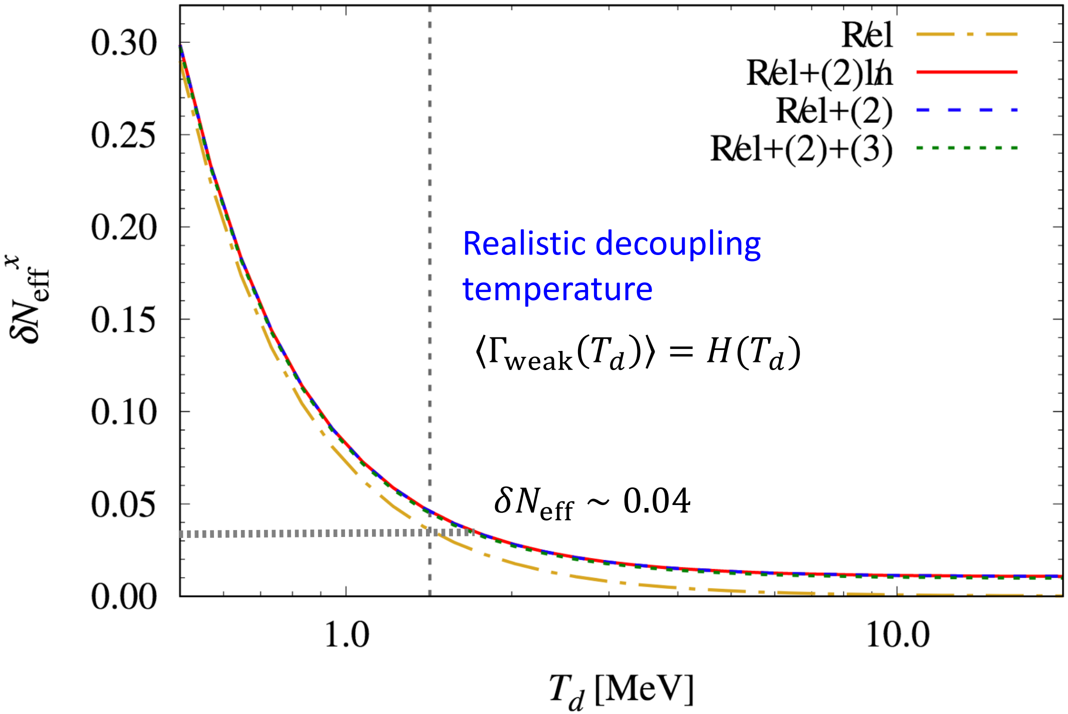
CνB-CMB temperature relation

Non-relativistic (m_e/T_d) correction to N_{eff}...

Change in N_{eff}:

- Even just allowing for a finite m_e in the QED plasma entropy will yield a large change $\delta N_{\mathrm{eff}} \sim 0.04$.
- This m_e/T_d correction to the CνB temperature is in fact the dominant correction to $N_{\mathrm{eff}}^{\mathrm{SM}}$.

Bennett, Buldgen, Drewes & Y^3W 2020



Realistic decoupling temperature

$\langle \Gamma_{\mathrm{weak}}(T_d) \rangle = H(T_d)$

$\delta N_{\mathrm{eff}} \sim 0.04$

Neutrino decoupling temperature

Non-ideal gas corrections from interactions…

The particle content and interactions at $0.1 < T < 10\ \mathrm{MeV}$ determine the properties of the CνB.

- **QED plasma**: $e^{\pm}, \gamma$
- **3 families of $\boldsymbol{\nu} + \bar{\boldsymbol{\nu}}$**: $\nu_e, \bar{\nu}_e,$ $\nu_\mu, \bar{\nu}_\mu,$ $\nu_\tau, \bar{\nu}_\tau$

EM interactions (always in equilibrium @ $0.1 < T < 10\ \mathrm{MeV}$):

$$e^+e^- \leftrightarrow \gamma\gamma$$
$$e^+e^- \leftrightarrow e^+e^-$$
$$e^{\pm}e^{\mp} \leftrightarrow e^{\pm}e^{\mp}$$
$$e^{\pm}e^{\pm} \leftrightarrow e^{\pm}e^{\pm}$$
$$\gamma e^{\pm} \leftrightarrow \gamma e^{\pm}$$

Deviations from an ideal gas described by thermal QED

Weak interactions (in equilibrium @ $T > O(1)\mathrm{MeV}$):

$$\nu_\alpha \nu_\beta \leftrightarrow \nu_\alpha \nu_\beta$$
$$\nu_\alpha \bar{\nu}_\beta \leftrightarrow \nu_\alpha \bar{\nu}_\beta$$
$$\bar{\nu}_\alpha \bar{\nu}_\beta \leftrightarrow \bar{\nu}_\alpha \bar{\nu}_\beta$$

$\alpha, \beta = e, \mu, \tau$

Coupled @ $T > O(1)\mathrm{MeV}$

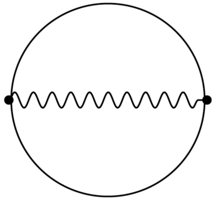
$$\nu_\alpha e^{\pm} \leftrightarrow \nu_\alpha e^{\pm}$$
$$\nu_\alpha \bar{\nu}_\alpha \leftrightarrow e^+e^-$$

Weak interactions (in equilibrium @ $T > O(1)\mathrm{MeV}$)

Finite-temperature QED…

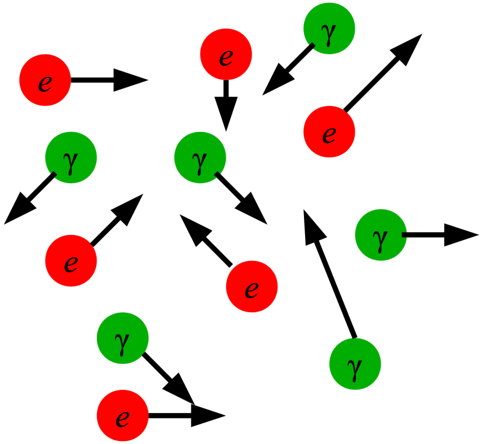
Lowest-order correction of the QED partition function

$$\ln Z^{(2)} = -\frac{1}{2}$$

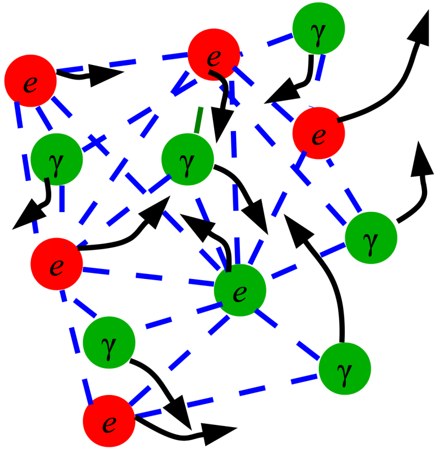


Interactions of $e^{\pm}, \gamma$ **modify the QED plasma** away from an ideal gas.

Ideal gas



+ EM interactions



T-dependent dispersion relation + Forces

Energy = kinetic energy + rest mass

Pressure = from kinetic energy

Energy = **modified** kinetic energy + **T-dependent masses** + **interaction potential energy**

Pressure = from **modified** kinetic energy + **EM forces**

➡ Modified QED equation of state

Finite-temperature effects on the QED EoS...

Finite-temperature corrections to the **QED partition function** at fixed order in the electric charge e are well known.

e.g., Kapusta textbook

- These can be easily implemented in the continuity equation to describe the energy density evolution in the QED sector:

$$\frac{d\rho_{\mathrm{QED}}}{dt} + 3H\left(\rho_{\mathrm{QED}} + P_{\mathrm{QED}}\right) = Q$$

Q = Energy exchange with the neutrino sector

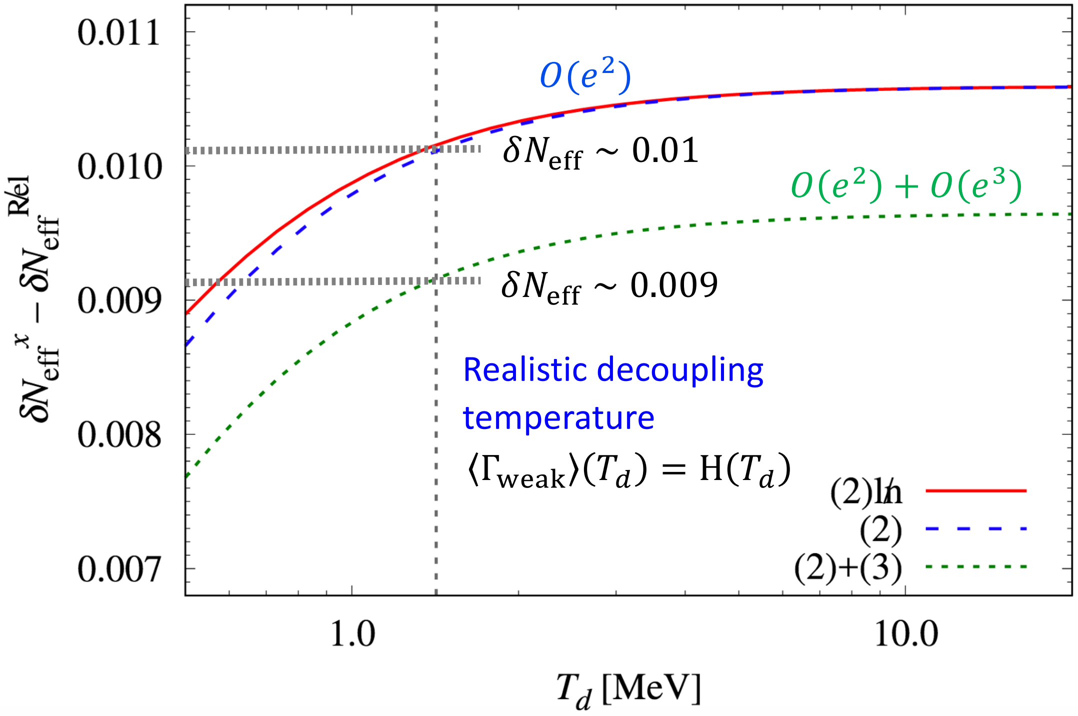
- The general outcome is a faster decrease of the photon temperature with expansion, leading to a **larger N_{eff}**.

FTQED EoS correction to N_{eff}…

Change in N_{eff}:

- Modified EoS causes the QED plasma to cool faster → Neutrinos appear hotter by $\delta N_{\mathrm{eff}} \sim 0.01$.
- $O(e^3)$ correction reduces the effect by $\delta N_{\mathrm{eff}} \sim -0.001$.
- Higher order and non-perturbative effects (bound state formation) $|\delta N_{\mathrm{eff}}| < 10^{-5}$.

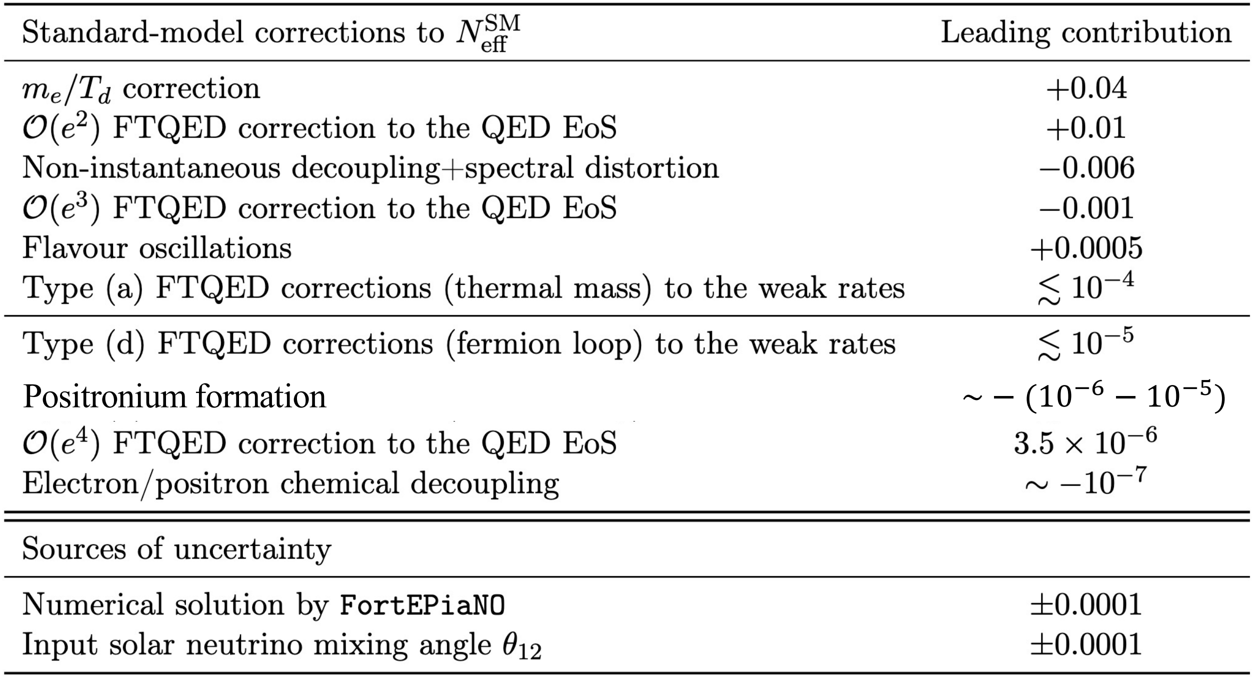
Bennett, Buldgen, Drewes & Y³W 2020



Neutrino decoupling temperature

Summary of corrections so far…

Leading contribution from various effects on $N_{\rm eff}^{\rm SM}$:

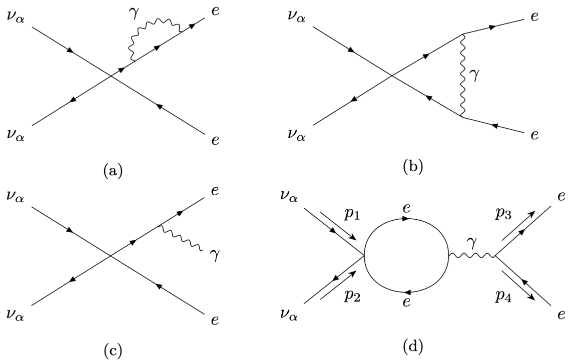


Standard-model corrections to $N_{\rm eff}^{\rm SM}$	Leading contribution
m_e/T_d correction	$+0.04$
$\mathcal{O}(e^2)$ FTQED correction to the QED EoS	$+0.01$
Non-instantaneous decoupling+spectral distortion	-0.006
$\mathcal{O}(e^3)$ FTQED correction to the QED EoS	-0.001
Flavour oscillations	$+0.0005$
Type (a) FTQED corrections (thermal mass) to the weak rates	$\lesssim 10^{-4}$
Type (d) FTQED corrections (fermion loop) to the weak rates	$\lesssim 10^{-5}$
Positronium formation	$\sim -(10^{-6} - 10^{-5})$
$\mathcal{O}(e^4)$ FTQED correction to the QED EoS	3.5×10^{-6}
Electron/positron chemical decoupling	$\sim -10^{-7}$
Sources of uncertainty	
Numerical solution by `FortEPiaNO`	± 0.0001
Input solar neutrino mixing angle θ_{12}	± 0.0001

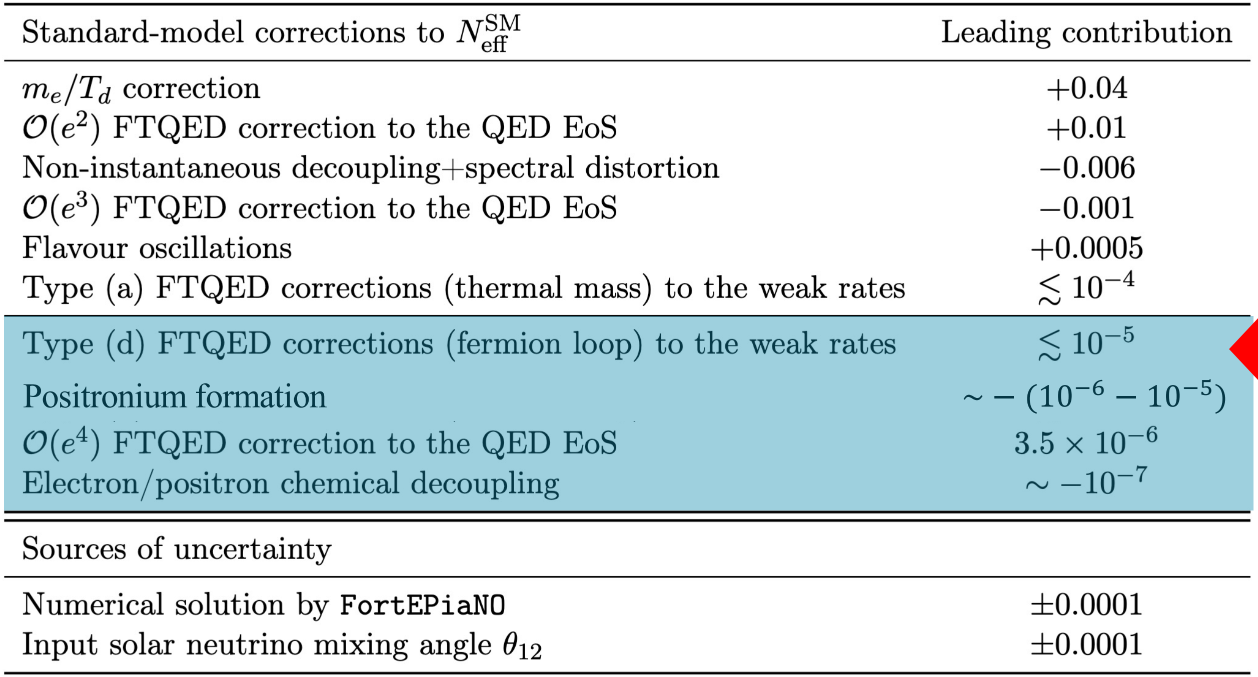
Accounted for in benchmark calculation

$$N_{\rm eff}^{\rm SM} = 3.0440 \pm 0.0002$$

Where to now?



Leading contribution from various effects on $N_{\text{eff}}^{\text{SM}}$:



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Accounted for in benchmark calculation

$$N_{\text{eff}}^{\text{SM}} = 3.0440 \pm 0.0002$$

Complete assessment of types (b-c) corrections; paper 5 (in prep)

Summary…

Cosmological observables such as the CMB and light element abundances can be very sensitive to light relics.

- Their energy density is quantified by their contributions to the "effective number of neutrinos" $N_{\mathrm{eff}} = N_{\mathrm{eff}}^{\mathrm{SM}} + \Delta N_{\mathrm{eff}}$, where $N_{\mathrm{eff}}^{\mathrm{SM}} \approx 3$ is the Standard-Model expectation.

- CMB measurements currently constrain N_{eff} to 10% uncertainty.
  - In the future, percent-level precision measurements are possible.

- In light of this potential improvement, we have performed a **new precision theoretical calculation** of the Standard Model expectation, $N_{\mathrm{eff}}^{SM} = 3.0440 \pm 0.0002$; the new value has already been implemented in the latest releases of `CLASS` and `CAMB`.